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# Chaos-assisted instanton tunneling in one-dimensional perturbed periodic potential

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For the system with a one-dimensional spatially periodic potential we demonstrate that small periodic in-time perturbation results in the appearance of chaotic instanton solutions. We estimate the parameter of local instability, the width of the stochastic layer, and the correlator for perturbed instanton solutions. The application of the instanton technique enables us to calculate the amplitude of the tunneling, the form of the spectrum, and the lower bound for the width of the ground quasienergy zone.

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## I. INTRODUCTION

Tunneling as an inherently quantum phenomenon attracts much attention [1]. Its connection with classical chaos in the semiclassical regime has also been discussed [2,3]. A number of works were devoted to semiclassical chaos assisted tunneling between symmetry-related Kolmogorov-Arnold-Moser (KAM) tori in systems with mixed dynamics (the well developed chaotic region coexists in phase space with regular islands) [4–6]. To describe chaos-assisted tunneling in systems with mixed dynamics the multilevel model Hamiltonian, primarily proposed in [2], is often used [7]. Less attention has been paid to semiclassical tunneling in KAM systems (the chaotic region is not widespread) [8]. Another way to describe semiclassical tunneling is based on solutions of Hamilton equations in imaginary time and path integral formalism [1]. The instanton technique [9] was used in a very few works [10].

In this work we consider one-dimensional quantum system with a periodic in-space potential affected by a small periodic in-time perturbation. We use methods created to describe chaos in classical Hamiltonian systems to investigate the essentially quantum phenomenon of tunneling. It is achieved in the framework of the instanton technique, where solutions of Euclidian equations of motion (instantons) play a dominating role with the use of standard methods from the viewpoint of chaos [11]. For the systems with periodic in-time perturbation energy is no more an exact integral of motion and the language of quasienergies is more adequate [12]. For some estimations energy as an adiabatic invariant can also be used [13]. We study properties of chaotic instanton solutions and calculate the form of the spectrum and the lower bound for the width of the ground quasienergy zone.

A Hamiltonian of the system under consideration is taken in the form

$$\tilde{H}=\frac{1}{2}\tilde{p}^2+\omega_0^2\cos x-\epsilon x\sum_{n=-\infty}^{+\infty}\delta(t-n\tilde{T}). \tag{1}$$

$\tilde{T}$ is the real-time period of perturbation, $\epsilon$ describes the strength of the perturbation. The mass of the particle equals the unit. The considered cosine potential corresponds to the nonlinear oscillator. The phase space of the nonlinear oscillator has the topology of the cylinder [points $(q,\tilde{p})$ and $(q+2\pi,\tilde{p})$ are identified]. Thus its (quasi)energy spectrum is discrete. Chaos-assisted tunneling between two major resonance islands (inside the single potential well) for driven nonlinear oscillator has been studied numerically in [4]. In this work we study the system with $x$ varying from $-\infty$ to $+\infty$. This results in the band structure of the (quasi)energy spectrum [14]. Systems with a spatially periodic potential were studied in instanton physics [15]. Perturbation used in Eq. (1) was exploited in the systems exhibiting quantum chaos [16,17].

There are papers devoted to chaos-assisted tunneling where some analytical predictions for billiard systems based on the path integral formulation of quantum mechanics are made [3]. A distinguishing feature of our work is the analytical predictions for the system with a smooth potential. For this purpose we explore the instanton technique adopted from quantum field theory [9].

## II. ANALYSIS OF CHAOTIC INSTANTON SOLUTIONS

For applying the instanton technique we consider solutions of classical equations of motion in *imaginary* (Euclidian) time. The Hamiltonian (1) has the same form (translated on $\pi$) in Euclidian time as in real time.

The Euclidian Hamiltonian of the system is $H=H_0+V$, where

$$H_0=\frac{1}{2}p^2-\omega_0^2\cos x \tag{2}$$

and

$$V=\alpha Tx\sum_{n=-\infty}^{+\infty}\delta(\tau-nT). \tag{3}$$

Here $H_0$ is the nonperturbed Euclidian Hamiltonian of the system and $V$ is the Euclidian potential of the perturbation.

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We also introduced the coupling constant $\alpha \ll 1$ instead of $\epsilon \equiv \alpha T$ in order to simplify formulas.

The nonperturbed instanton solution describes the motion on the separatrix of the Hamiltonian (2). It is well known that this separatrix is destroyed by any periodic perturbation and on its place a stochastic layer is present [13]. Perturbed instanton solutions correspond to the motion in the vicinity of the separatrix inside the layer. Therefore, instead of one instanton solution connecting the neighbor maxima of the nonperturbed Euclidian potential (classical vacuum states in a real-time potential), we obtain a manyfold of instanton solutions of Euclidian equations placed inside the stochastic layer.

We calculate the parameter of the local instability, the width of the stochastic layer, and the correlator for perturbed instanton solutions. It is convenient to describe the dynamics of the system in action-angle variables [11]. The equation of motion for the action variable has the form

$$\dot{I} = -\frac{\alpha \dot{x}}{\omega}\left(2\sum_{m=1}^{+\infty}\cos(m\nu\tau)+1\right). \tag{4}$$

Here $\omega(I) \equiv dH_0/dI$ is the nonlinear frequency [11]. Instead of an angle variable we introduce a phase of the external force $\psi$ defined by the relation $\dot{\psi}=\nu\equiv 2\pi/T$ [13]. Let $H_s \equiv \omega_0^2$ denote the energy of a nonperturbed system on the separatrix. Continuous equations of motion for $I$ and $\psi$ can be reduced to discrete mapping for the phase of external force in the vicinity of the separatrix ($|H-H_s| \ll 1$) [13,16,17]

$$\psi_{n+1} = \psi_n + B_n + K_0 \sin \psi_n, \tag{5}$$

where

$$K_0 = \frac{8\pi\alpha\nu}{\omega_0}\frac{e^{-\pi\nu/2\omega_0}}{|H-H_s|},$$

$B_n$ are some functions of $H$ whose exact form is not essential for our purposes. We assume, following [13], that due to a small value of perturbation the energy practically does not change with time and equals the energy of the nonperturbed system. The map (5) with an arbitrary parameter $K_0$ was studied by many authors, for instance [16]. Particulary, it is known that at $K_0 > 1$ the motion is locally unstable and chaotic, whereas at $K_0 \ll 1$ it is stable and regular. Thus $K_0$ is the parameter of local instability. Condition $K_0 \sim 1$ enables us to calculate the width of the stochastic layer

$$|H_s - H_b| = \frac{8\pi\alpha\nu}{\omega_0}e^{-\pi\nu/2\omega_0}. \tag{6}$$

Here $H_b$ is the energy value on the bound of the stochastic layer.

To calculate the correlator for the perturbed instanton solutions we use the standard technique [18]. For the map (5) the correlator is

$$R(\tau,\tau_0) = \frac{1}{2\pi}\int_0^{2\pi} d\psi_0 \exp\{i[\psi(\tau)-\psi_0]\} \sim \exp\left(-\frac{\tau-\tau_0}{\tau_R}\right). \tag{7}$$

Here $\psi_0 \equiv \psi(\tau_0)$ and the time of correlations decay is $\tau_R = 2\pi/(\omega \ln K_0)$. The exponential decrease of the correlator shows that the dynamics of the instanton solutions inside the stochastic layer ($K_0 > 1$) possesses the property of mixing (chaos) [11].

Note that the perturbed one-instanton solution due to the stochastic layer connects not only neighbor vacua of real-time potential but also two *arbitrary* chosen vacua. We note that in order to describe tunneling between non-neighbor vacua of nonperturbed systems one has to take into account the contribution of multi-instanton configurations [15].

## III. THE CALCULATION OF THE TUNNELING AMPLITUDE AND GROUND ZONE WIDTH

Let us consider the tunneling between neighbor vacua (from $x \approx -\pi$ to $x \approx \pi$ for distinctness) in the presence of perturbation (3). In Euclidian time this tunneling process for the nonperturbed system is described by the solution of Euclidian equations of motion with asymptotes $x=-\pi$, $p=0$ at $\tau=-\infty$ and $x=\pi$, and $p=0$ at $\tau=+\infty$. There is only one solution satisfying these conditions for the nonperturbed system (2) (a one-instanton solution),

$$x_0^{inst}(\tau-\tau_0) = -\pi + 4\arctan e^{\omega_0(\tau-\tau_0)}. \tag{8}$$

Its Euclidian action is $S^{inst} = 8\omega_0$. The instanton's position is denoted by $\tau_0$. Due to Euclidian equations of motion and the antisymmetry of $x_0^{inst}$, when time is inverted with respect to the point $\tau_0$ perturbation (3) does not change the Euclidian action of the one-instanton solution (8) in the first order on the coupling constant $S_{pert}^{inst} = S^{inst} + O(\alpha^2)$. The only manifestation of the perturbation in this approximation is the appearance of a number of the additional solutions of Euclidian equations of motion with energies close to the energy of the nonperturbed one-instanton solution and placed inside the stochastic layer.

Let us consider firstly the nonperturbed system at arbitrary energy $-\omega_0^2+\varepsilon$, $0<\varepsilon<2\omega_0^2$. One-half of the truncated instanton action can be easily calculated,

$$S[x^{inst}(\tau,\varepsilon)] = \int_{-a(\varepsilon)}^{a(\varepsilon)} \sqrt{2[\omega_0^2\cos x - (-\omega_0^2+\varepsilon)]}\,dx$$

$$= 4\sqrt{4\omega_0^2 - 2\varepsilon}\,E\left(a(\varepsilon)\frac{1}{1-\dfrac{\varepsilon}{2\omega_0^2}}\right), \tag{9}$$

where $\pm a(\varepsilon) = \pm \arcsin\sqrt{1-\varepsilon/2\omega_0^2}$ are turning points, and the function $E$ is the elliptic integral of the second kind.

Then the tunneling amplitude in the *perturbed* system can be found by integration over the energy of the tunneling amplitude in the *nonperturbed* system with the action (9)

$$A=\int_0^{\Delta H} d\varepsilon \int_{x(\tau)\approx-\pi}^{x(\tau)\approx\pi} Dx\ \exp\{-S[x^{inst}(\tau,\varepsilon)]\}, \quad (10)$$

where $\Delta H=2|H_s-H_b|$ is the stochastic layer width. The contribution of the chaotic instanton solutions is taken into account by means of integration over $\varepsilon$. Expression (10) shows that the probability of tunneling (the square of the absolute value of the tunneling amplitude) grows while the chaotic region spreads ($\Delta H$ increases).

The result is obtained in the first order on the coupling constant $\alpha$ and does not take into account the possible structure of the stochastic layer. It is valid if the layer is narrow and is in agreement with results of numerical [19,20] and real [6] experiments for similar problems. We also have correspondence in Eq. (10) with the nonperturbed case [15]. Namely, if $\alpha=0$ then $\Delta H=0$ and the single solution describing the motion on the separatrix (the nonperturbed one-instanton solution) contributes to the tunneling amplitude.

Formula (10) can be made more transparent if we use the approximate form of action (9) at $\varepsilon<2\omega_0^2$,

$$S[x^{inst}(\tau,\varepsilon)]\approx 8\omega_0-\frac{\pi\varepsilon}{\omega_0}. \quad (11)$$

Then in the Gauss approximation we obtain the following expression for the tunneling amplitude:

$$A=\int_0^{\Delta H} d\varepsilon \int_{-\infty}^{+\infty} dc_0 \sqrt{S[x^{inst}(\tau,\varepsilon)]}\exp\{-S[x^{inst}(\tau,\varepsilon)]\}$$
$$\approx e^{-S^{inst}}\sqrt{S^{inst}}\Gamma F=\sqrt{8\omega_0}\Gamma e^{-8\omega_0}e^{\pi\Delta H/\omega_0}, \quad (12)$$

where integration over $c_0$ gives the contribution of zero modes, and $\Gamma$ is a time of the tunneling. Formula (12) up to the factor *i* has the same form in real Minkovski time.

Expression (12) can be interpreted in the following way. Factor $F=\exp(\pi\Delta H/\omega_0)>1$ in Eq. (12) differs between perturbed and nonperturbed amplitudes and includes the contribution of the layer. One can think about $F$ as a number of instanton solutions inside the stochastic layer.

We can find the minimal number of instanton solutions inside the stochastic layer. One cannot distinguish instanton solutions within the energy interval $\Delta E\sim 1/\Delta\tau$ (the Heisenberg uncertainty relation). Here $\Delta\tau\sim\omega_0^{-1}$ denotes the time interval of observation. Thus the energy interval between the neighbor instantons is $\Delta\varepsilon\sim\omega_0$. Therefore parameter $F$ can be found as follows:

$$F\sim 1+\frac{\Delta H}{\Delta\varepsilon}\approx e^{\Delta H/\omega_0}, \quad (13)$$

where the unit takes into account the nonperturbed separatrix.

The form of the spectrum of the lower quasienergy zone is obtained using the amplitude (12) by means of the standard technique (multi-instanton contributions are taken into account) [15],

$$E_\theta\approx\frac{1}{2}\omega_0-2e^{-S^{inst}}\sqrt{S^{inst}}F\cos\theta. \quad (14)$$

Here the continuous variable $\theta$ parametrizes levels of the ground quasienergy zone. The zone width

$$\Delta E\approx 4e^{-S^{inst}}\sqrt{S^{inst}}F \quad (15)$$

differs from the nonperturbed case by a factor $F$ that reflects the influence of perturbation.

## IV. CONCLUSION

We applied the theory of *classical* chaos for an investigation of the chaos-assisted *tunneling* in terms of the path integral formalism in imaginary time and instanton techniques. We found the parameters of the local instability and the width of the stochastic layer. An exponential decrease of the correlator for any perturbed instanton solution was also demonstrated, which means it possesses the property of mixing. Then properties of the stochastic layer and classical chaotic solutions in Euclidean space (chaotic instantons) were used for the calculation tunneling amplitude and the ground quasienergy zone spectrum in the presence of the perturbation and the zone width.

The general tendency for the chaos-assisted tunneling regime (on average — if we abstract from fluctuations) is an increase of the tunneling amplitude (probability) as the strength of the perturbation increases [6,20]. It is confirmed here. The reason is the growth of the width of the chaotic layer and therefore the increase of the number of paths for a particle to travel from one regular region to another. For small energies in the Gauss approximation the tunneling amplitude is increased by the factor $F>1$. The lifetime of the particle in a certain vacuum of the system decreases. It is connected with the widening of the (quasi)energy zone (15).

We would like to emphasize that the obtained results are not consequences of a particular choice of the nonperturbed Hamiltonian (2) or perturbation (3). Qualitatively they are valid for a more general class of one-dimensional nonperturbed Hamiltonians with a quadratic dependence on the momentum and the spatially periodic potential with a single well in each period, as well as the fact that the time dependence of the homogeneous perturbation can be realized by any time periodic function. The reason is the universality of the separatrix destruction mechanism in these potentials that is affected by time-periodic perturbation [13].

Tunneling plays an important role in gauge field theories (instanton physics [9]). The experimental discovery of QCD instantons, for example, is an important problem [21]. Moreover, it is known that classical gauge field theories are inherently chaotic [22]. Therefore, the study of chaos-assisted instanton tunneling in gauge field theories based on chaos criterion in quantum field theory [23] is also of essential interest.

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